# Math 429 - Exercise Sheet 9

**1.** We already saw the root decomposition of $\mathfrak{sl}_n$ (it is called "type $A$"). Work out the root decomposition of $\mathfrak{g} \in \{\mathfrak{o}_{2n+1}, \mathfrak{sp}_{2n}, \mathfrak{o}_{2n}\}$ (which are called "types $B$,$C$,$D$", respectively) by starting from the Cartan subalgebra of $\mathfrak{g}$ consisting of diagonal matrices.

**2.** Compute explicitly the $\mathfrak{sl}_2$-triples associated to the root decompositions of $\mathfrak{o}_{2n+1}, \mathfrak{sp}_{2n}, \mathfrak{o}_{2n}$ you found in the previous part.

**3.** Show that any 3-dimensional complex semisimple Lie algebra is isomorphic to $\mathfrak{sl}_2$.

**4.** Show that the root spaces of any complex semisimple Lie algebra $\mathfrak{g}$ satisfy

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$$

whenever the roots $\alpha, \beta$ satisfy $\alpha + \beta \neq 0$ (the inclusion $\subseteq$ is obvious, it's $\supseteq$ that's tricky).

**5.** Using the result below as a black box, show that indeed the elements $H_\alpha$ defined in Lecture do not depend on the choice of s.i.b.f. on a complex semisimple Lie algebra $\mathfrak{g}$.

**(*)** If $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ with $\mathfrak{g}_1, \ldots, \mathfrak{g}_k$ simple, show that any Cartan subalgebra in $\mathfrak{g}$ is the direct sum of a collection of Cartan subalgebras in $\mathfrak{g}_1, \ldots, \mathfrak{g}_k$. In this case, also conclude that the root decomposition of $\mathfrak{g}$ is the corresponding direct sum of the root decompositions of $\mathfrak{g}_1, \ldots, \mathfrak{g}_k$.